General Kaluza-Klein black holes with all six independent charges in five-dimensional minimal supergravity

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Using the $SL(2,R)$-duality in a dimensionally reduced spacetime in (the bosonic sector of) five-dimensional minimal supergravity, we construct general Kaluza-Klein black hole solutions which carry six independent charges, its mass, angular momentum along four dimensions, electric and magnetic charges of the Maxwell fields in addition to Kaluza-Klein electric and magnetic monopole charges.

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I. INTRODUCTION

In string theory and various related contexts, higher dimensional black holes have played an important role. In particular, physics of black holes in the five-dimensional Einstein-Maxwell-Chern-Simons (EMCS) theory has recently been the subject of increased attention, as the EMCS theory describes the bosonic sector of five-dimensional minimal supergravity, a sub-sector of a low-energy limit of string theory. The dimensional reduction of minimal supergravity to four dimensions yields two Maxwell fields, a massless axion and a dilaton, all coupled to gravity [1], where (as was shown in Ref. [2],) the equations of motions derived from the dimensional reduction are invariant under the action

Solutions in $D=5$ minimal supergravity	M	J	Q	P	q	p
Gaiotto et al. [3]	yes†	no	yes	yes†	yes†	no
Elvang et al. [4]	yes†	no	yes	yes†	yes†	yes†
Ishihara-Matsuno [5]	yes	no	no	yes	yes	no
Nakagawa et al. [6]	yes	no	yes†	yes†	yes†	yes†
Tomizawa et al. [7]	yes	no	yes	yes	yes	yes
Tomizawa et al. [8]	yes	yes	no	yes	yes	no
Compere et al. [9]	yes	yes	yes	no	yes	yes
Mizoguchi-Tomizawa [10]	yes	yes	yes†	yes†	yes†	yes†

TABLE I: Classification of Kaluza-Klein black holes in five-dimensional minimal supergravity: The six charges, M, J, Q, P, q and p denote, respectively, their mass, angular momentum, Kaluza-Klein electric charge, Kaluza-Klein magnetic charge, electric charge and magnetic charge. Here the charges with a dagger " † " for each solution are not independent but related by a certain constraint.

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of a global $SL(2,R)$ group, by which the Maxwell fields are related to the Kaluza-Klein electromagnetic fields. This so-called $SL(2,R)$-duality enables us to generate a new solution in (the bosonic sector of) five-dimensional minimal supergravity by stating from a certain known solution in the same theory.

Known Kaluza-Klein black hole solutions of five-dimensional minimal supergravity are summarized in TABLE I, where they are classified by their conserved charges (from the four-dimensional perspective) : mass, angular momentum, Kaluza-Klein electric/magnetic charges and electric/magnetic charges of the Maxwell field. As shown in the list, the most general black hole solutions with full six independent charges, which are expected to exist [11], have not been discovered so far. The aim of this paper is to present such exact solutions describing general Kaluza-Klein black holes with full six charges obtained by using our framework [10, 12] of the $SL(2,R)$-duality.

In our previous work [10], applying the $SL(2,R)$-duality symmetry to the Rasheed solutions [13], which are known to describe dyonic rotating black holes (from the four-dimensional point of view) of five-dimensional pure gravity, we obtained six-charge rotating Kaluza-Klein black hole solutions in five-dimensional minimal supergravity. However, it turns out that four (Kaluza-Klein electric/magnetic charges and electric/magnetic charges of the Maxwell field) of these conserved charges are related by a constraint, namely, these parameters are not wholly independent. In this paper, we alternatively use as a seed solution the boosted rotating (electrically/magnetically charged) black string solutions with five parameters obtained in Ref. [9]. As is well known, the boost along the fifth dimension yields an (Kaluza-Klein's) electric charge in the dimensionally reduced four-dimensional theory. Therefore, our starting-point solutions have five charges (mass, angular momentum along four dimensions, the Kaluza-Klein electric charge and electric/magnetic charges of the Maxwell field). The $SL(2,R)$-duality transformation then adds the missing Kaluza-Klein monopole charge to the seed solution.

The remainder of this paper is organized as follows: In the next section, we present the metric and gauge potential 1-form of the Maxwell field of the seed solution, which is our starting point. In Section III, by acting the $SL(2,R)$ transformation on the seed solution, we derive the most general Kaluza-Klein black hole solutions in the above sense, and write down the metric and Maxwell fields. In Section IV, we show that our solution describes rotating black holes with six conserved charges (mass, angular momentum, Kaluza-Klein electric/magnetic charges and electric/magnetic charges of the Maxwell field) in the dimensionally reduced four-dimensional theory. In Section V, we discuss limits of our solution to some known ones. Section VI is devoted to summarizing our results. The new solution contains complicated polynomials of r (the radial coordinate) and $x=\cos\theta$ (with θ being an angle coordinate), and their coefficients are collected in Appendix A. Finally, Appendix B is a brief summary of the $SL(2,R)$ duality.

II. SEED SOLUTION

In this paper, we choose the black string solutions with five independent parameters as a seed solution in Ref [9], whose metric and gauge potential 1-form are given by, respectively,

$$\begin{aligned} ds^2 &= \frac{\Sigma}{\xi(F_1+\Delta_2)^2}\left(dx^5+B_\mu dx^\mu\right)^2 \\ &\quad +\frac{\xi^{1/2}(F_1+\Delta_2)}{\Sigma^{1/2}}\left[-\frac{\xi^{1/2}\Delta_2}{\Sigma^{1/2}}\left(dt+\omega d\phi\right)^2+\frac{\Sigma^{1/2}}{\xi^{1/2}}\left(\frac{\Delta(1-x^2)}{\Delta_2}d\phi^2+\frac{dr^2}{\Delta}+\frac{dx^2}{1-x^2}\right)\right], \end{aligned} \tag{1}$$

$$\begin{aligned} A &= -\frac{\sqrt{3}(c_gF_3+s_gF_2)}{F_1+\Delta_2}dt+\left[\frac{2\sqrt{3}mc_d^2}{\Delta_2}\left\{-2s_bc_bf\Delta x+as_d(2ms_b^2(6c_b^2c_d^2f^2-1)+(c_b^2+s_b^2)r)(1-x^2)\right\}\right. \\ &\quad \left.-\frac{\sqrt{3}}{F_1+\Delta_2}\left\{\frac{4mas_bc_bc_d^3f(r+2s_b^2m)}{\Delta_2}(1-x^2)+F_3\omega_\phi\right\}\right]d\phi-\frac{\sqrt{3}(c_gF_2+s_gF_3)}{F_1+\Delta_2}dx^5, \end{aligned} \tag{2}$$

where

$$\begin{aligned} B_\mu dx^\mu &= \frac{-c_gs_g\Delta_2(F_1+\Delta_2)^3+(s_gk+c_g\xi)(c_gk+s_g\xi)}{\Sigma}dt \\ &\quad +\frac{k\omega_\phi(ks_g+c_g\xi)-s_g\Delta_2(F_1+\Delta_2)^3\omega_\phi+\frac{(s_gk+c_g\xi)(4mac_bs_bc_d^3(r+2ms_b^2))(1-x^2)}{\sqrt{1+3c_d^2}\Delta_2}}{\Sigma}d\phi, \end{aligned} \tag{3}$$

$$\omega=c_g\omega_\phi+s_g\frac{-4mac_bs_bc_d^3(r+2ms_b^2)}{\sqrt{1+3c_d^2}\Delta_2}(1-x^2), \tag{4}$$

$$\omega_\phi = -2mac_d^3\frac{1-x^2}{\Delta_2}\left[2ms_b^2\left(c_b^2+s_b^2-\frac{4c_b^2}{1+3c_d^2}\right)+r(c_b^2+s_b^2)\right], \tag{5}$$

$$\Sigma = (s_g k + c_g\xi)^2 - s_g^2\Delta_2(F_1+\Delta_2)^3, \tag{6}$$

$$\Delta = r^2 - 2mr + a^2, \quad \Delta_2 = r^2 - 2mr + a^2x^2, \tag{7}$$

$$\xi = (F_4+\Delta_2)(F_1+\Delta_2) - F_2^2, \quad k = F_5(F_1+\Delta_2) - F_2F_3, \tag{8}$$

$$F_1 = 2mc_d^2\left[2ms_b^2(s_b^2+s_d^2c_b^2f^2)+(c_b^2+s_b^2)r+2as_ds_bc_bfx\right], \tag{9}$$
$$F_2 = 2ms_dc_d\left[2ms_bc_bf(-1+(c_b^2+s_b^2)c_d^2)+2s_bc_bfr+as_d(c_b^2+s_b^2)x\right], \tag{10}$$
$$F_3 = 2mc_d\left[2ms_ds_b^2(c_b^2(1+c_d^2)f^2-s_b^2)-s_d(c_b^2+s_b^2)r+2as_bc_bfx\right], \tag{11}$$
$$F_4 = 2m\left[2m((c_b^2s_d^2+s_b^2c_d^2)^2+s_d^2s_b^2c_b^2f^2)+(c_b^2+s_b^2)(c_d^2+s_d^2)r-2as_bc_bs_dfx\right], \tag{12}$$
$$F_5 = 2m\left[2ms_bc_bf(-1+(c_b^2+s_b^2)c_d^4)+2s_bc_bfr-as_d^2(c_b^2+s_b^2)x\right], \tag{13}$$

$$s_b = \sinh b,\ c_b = \cosh b,\ s_d = \sinh d,\ c_d = \cosh d,\ s_g = \sinh g,\ c_g = \cosh g, \tag{14}$$
$$f = \frac{1}{\sqrt{1+3c_d^2}}. \tag{15}$$

Here, we would like to note that our definition for the function Σ is different from one in Ref. [9]. As will be seen later, the familiar parameters m and a denote the mass and rotational parameter, respectively, and three parameters, b and d and g, correspond to the magnetic charge, electric charge of the Maxwell $U(1)$ gauge field and electric charge of the Kaluza-Klein $U(1)$ gauge field, respectively.

III. TRANSFORMATION

Now, in order to construct six-charge solutions, we apply the $SL(2,R)$-duality transformation to the above solution. Some necessary transformation formula developed in our previous work [10] are briefly summarized in Appendix B. From Eqs. (1) and (2), one can read off the dilaton and axion for the seed as

$$\rho = \frac{\Sigma^{1/2}}{\sqrt{\xi}(F_1+\Delta_2)}, \quad A_5 = -\sqrt{3}\frac{s_gF_3+c_gF_2}{F_1+\Delta_2}. \tag{16}$$

Therefore, from Eqs. (B12) and (B13) in Appendix B, the dilaton and axion fields for the transformed solutions are written as, respectively,

$$\rho_{new} = \frac{\Sigma^{1/2}\xi^{1/2}(F_1+\Delta_2)}{\Pi^2+\beta^2\Sigma}, \tag{17}$$

$$A_5^{new} = \sqrt{3}\xi^{1/2}\frac{\Pi[\alpha(F_1+\Delta_2)-(cF_2+sF_3)]+\beta\Sigma}{\Pi^2+\beta^2\Sigma}, \tag{18}$$

where the function Π is

$$\Pi = \xi^{1/2}[\gamma(F_1+\Delta_2)-\beta(cF_2+sF_3)]. \tag{19}$$

On the other hand, from Eqs. (B14) and (B15) in Appendix B, the gauge potential 1-forms of Kaluza-Klein's and Maxwell's $U(1)$ gauge fields for the transformed solutions are written as

$$B_\mu^{new} = \sqrt{3}\beta^2\gamma\tilde{A}_\mu + \left(\gamma^3+\sqrt{3}\beta\gamma^2A_5\right)B_\mu - \sqrt{3}\beta\gamma^2A_\mu + \beta^3\tilde{B}_\mu, \tag{20}$$

$$A_\mu^{new} = \left[\sqrt{3}\beta^2\gamma A_5^{new} - \beta(2+3\alpha\beta)\right]\tilde{A}_\mu + \left[-\sqrt{3}\alpha\gamma^2 + \gamma^3 A_5^{new} - A_5((1+4\alpha\beta+3\alpha^2\beta^2) - \sqrt{3}\beta\gamma^2 A_5^{new}))\right]B_\mu$$
$$+\left[(1+4\alpha\beta+3\alpha^2\beta^2) - \sqrt{3}\beta\gamma^2 A_5^{new})\right]A_\mu + \left[-\sqrt{3}\beta^2 + \beta^3 A_5^{new}\right]\tilde{B}_\mu, \tag{21}$$

where the 1-forms $\tilde{B}_\mu dx^\mu$ $(\mu = t, \phi)$ and $\tilde{A}_\mu dx^\mu$ $(\mu = t, \phi)$ can be obtained by integrating Eqs. (B16) and (B6) in Appendix B, and they are explicitly written as

$$\tilde{B}_t = \frac{(a_1+a_2x)r^2 + (a_3+a_4x)r + a_5 + a_6x + a_7x^2 + a_8x^3}{\Gamma}, \tag{22}$$

$$\tilde{B}_\phi = c_1 + c_2x + (1-x^2)\frac{b_1r^3 + (b_2+b_3x)r^2 + (b_4+b_5x+b_6x^2)r + b_7 + b_8x + b_9x^2 + b_{10}x^3}{\Gamma}, \tag{23}$$

$$\tilde{A}_t = \frac{p_1r^3 + (p_2+p_3x)r^2 + (p_4+p_5x+p_6x^2)r + p_7 + p_8x + p_9x^2 + p_{10}x^3}{\Gamma}, \tag{24}$$

$$\tilde{A}_\phi = r_1 + r_2x + (1-x^2)\frac{q_1r^3 + (q_2+q_3x)r^2 + (q_4+q_5x+q_6x^2)r + q_7 + q_8x + q_9x^2 + q_{10}x^3}{\Gamma}, \tag{25}$$

where the function Γ is

$$\Gamma = \frac{-s^2\Delta_2(F_1+\Delta_2)^3 + (c_g\xi + s_gk)^2}{\xi}, \tag{26}$$

and the constants $a_1, \cdots, a_8, b_1, \cdots, b_{10}, c_1, c_2, p_1, \cdots, p_{10}, q_1, \cdots, q_{10}, r_1, r_2$ are related to the five-parameters m, a, b, d and g only (the explicit forms are given in Appendix A). Note that the constant c_1 can be set to be zero by a coordinate transformation.

IV. MOST GENERAL SOLUTIONS

From the previous section, we can get six-charge Kaluza-Klein solutions in the same theory, whose metric and gauge potential 1-form are written as, respectively,

$$ds^2 = \frac{\Sigma\xi(F_1+\Delta_2)^2}{(\Pi^2+\beta^2\Sigma)^2}\left[dx^5 + \left\{\sqrt{3}\beta^2\gamma\tilde{A}_\mu + \left(\gamma^3 + \sqrt{3}\beta\gamma^2 A_5\right)B_\mu - \sqrt{3}\beta\gamma^2 A_\mu + \beta^3\tilde{B}_\mu\right\}dx^\mu\right]^2$$
$$+\frac{\Pi^2+\beta^2\Sigma}{\Sigma^{1/2}\xi^{1/2}(F_1+\Delta_2)}\left[-\frac{\xi^{1/2}\Delta_2}{\Sigma^{1/2}}(dt+\omega d\phi)^2 + \frac{\Sigma^{1/2}}{\xi^{1/2}}\left(\frac{\Delta}{\Delta_2}(1-x^2)d\phi^2 + \frac{dr^2}{\Delta} + \frac{dx^2}{1-x^2}\right)\right], \tag{27}$$

$$A^{new} = \left[\left\{\sqrt{3}\beta^2\gamma A_5^{new} - \beta(2+3\alpha\beta)\right\}\tilde{A}_\mu + \left\{-\sqrt{3}\alpha\gamma^2 + \gamma^3 A_5^{new} - A_5((1+4\alpha\beta+3\alpha^2\beta^2) - \sqrt{3}\beta\gamma^2 A_5^{new}))\right\}B_\mu\right.$$
$$\left.+\left\{(1+4\alpha\beta+3\alpha^2\beta^2) - \sqrt{3}\beta\gamma^2 A_5^{new}\right\}A_\mu + \left\{-\sqrt{3}\beta^2 + \beta^3 A_5^{new}\right\}\tilde{B}_\mu\right]dx^\mu$$
$$+\left[\sqrt{3}\xi^{1/2}\frac{\Pi[\alpha(F_1+\Delta_2) - (c_gF_2 + s_gF_3)] + \beta\Sigma}{\Pi^2+\beta^2\Sigma}\right]dx^5. \tag{28}$$

V. CHARGES

In this theory, the electric/magnetic charges (Q/P) of the Kaluza-Klein $U(1)$ field and electric/magnetic charges (q/p) of the Maxwell field are defined by, respectively,

$$Q = \frac{1}{8\pi}\int_{S^2}\mathcal{H}^B,\ P = \frac{1}{8\pi}\int_{S^2}\mathcal{B}, \tag{29}$$

$$q = \frac{1}{8\pi}\int_{S^2}\tilde{\mathcal{A}},\ p = \frac{1}{8\pi}\int_{S^2}\mathcal{F}'. \tag{30}$$

where S^2 denotes any closed two-surfaces surrounding the black hole. The two form fields $\mathcal{H}^B$ and $\mathcal{B}$ are defined by $\mathcal{H}^B := \frac{1}{2}H^B_{\mu\nu}dx^\mu \wedge dx^\nu$ and $\mathcal{B} := \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu$, respectively. Similarly, where $\tilde{\mathcal{A}} := \frac{1}{2}\tilde{A}_{\mu\nu}dx^\mu \wedge dx^\nu$ and $\mathcal{F}' := \frac{1}{2}F'_{\mu\nu}dx^\mu \wedge dx^\nu$. The four charges for the new solution are related to those of seed by

$$\begin{pmatrix} q^{new} \\ P^{new} \\ -p^{new} \\ Q^{new} \end{pmatrix} = \begin{pmatrix} 1+3\alpha\beta & \sqrt{3}\alpha^2(1+\alpha\beta) & \alpha(2+3\alpha\beta) & \sqrt{3}\beta \\ \sqrt{3}\beta^2(1+\alpha\beta) & (1+\alpha\beta)^3 & \sqrt{3}\beta(1+\alpha\beta)^2 & \beta^3 \\ \beta(2+3\alpha\beta) & \sqrt{3}\alpha(1+\alpha\beta)^2 & 1+4\alpha\beta+3\alpha^2\beta^2 & \sqrt{3}\beta^2 \\ \sqrt{3}\alpha & \alpha^3 & \sqrt{3}\alpha^2 & 1 \end{pmatrix} \begin{pmatrix} q \\ 0 \\ -p \\ Q \end{pmatrix}, \tag{31}$$

where Q and q/p are the electric charge for Kaluza-Klein $U(1)$ field and electric/magnetic charge for Maxwell field for the seed solution can be explicitly written as

$$Q = m\left[(1+3s_d^2)(c_b^2+s_b^2)s_g c_g + 2s_b c_b(c_g^2+s_g^2)f\right], \tag{32}$$

$$q = \sqrt{3}ms_d c_d\left[(c_b^2+s_b^2)c_g - 2s_b c_b s_g f\right], \tag{33}$$

$$p = 2\sqrt{3}ms_b c_b c_d^2 f. \tag{34}$$

Note that for the seed (boosted black string), the Kaluza-Klein's magnetic monopole charge P vanishes. One of the two parameters α and β corresponds to adding a constant to A_5, which means that gauge transformation for the potential 1-form A_M. Therefore, the remaining one corresponds to adding a physical degree of freedom, i.e., a Kaluza-Klein monopole charge.

In our previous work [10], we derived a different six-charge solution starting from a different seed solution, but the charges of that solution were not wholly independent, but four of them (P,Q,p,q) were related by a certain constraint. Here we would like to confirm whether or not these conserved charges are actually independent. As is easily verified, the following Jacobian does not vanish

$$\frac{\partial(Q^{new}, P^{new}, q^{new}, p^{new})}{\partial(b,d,g,\beta)}, \tag{35}$$

which therefore means that these charges are independent of one another. Also, it is clear that the mass M and angular momentum J are not related to these four charges.

VI. SOME LIMITS

Here, we study some limits to known Kaluza-Klein black hole solutions by taking some parameter limits in our solutions. First, we take the limit of $\alpha = -1, \beta = 1,\ a = 0,\ d = 0$. Defining the parameters $(\varrho_\pm, \varrho_0)$ by

$$\varrho_- = 2ms_b^2,\ \varrho_+ = 2mc_b^2,\ \varrho_0 = 2ms_g(s_g + 2s_b^2 s_g + 2s_b c_b c_g), \tag{36}$$

and the coordinates (ϱ, θ, ψ) by

$$r = \varrho + m(1 - c_b^2 - s_b^2),\ x = -\cos\theta,\ x^5 = 2m(s_g c_g + 2s_b^2 s_g c_g + s_b c_b(c_g + s_g))\psi, \tag{37}$$

one can obtain the metric:

$$\begin{aligned} ds^2 &= -\frac{(\varrho-\varrho_+)(\varrho-\varrho_-)}{\varrho^2}dt^2 + \frac{\varrho(\varrho+\varrho_0)}{(\varrho-\varrho_+)(\varrho-\varrho_-)}d\varrho^2 + \varrho(\varrho+\varrho_0)(d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad + \frac{\rho(\varrho_+ + \varrho_0)(\varrho_- + \varrho_0)}{\varrho+\varrho_0}(d\psi + \cos\theta d\phi)^2. \end{aligned} \tag{38}$$

This is the metric of the Ishihara-Matsuno solutions [5].

Next, we identify the coordinate and the parameters as $\varrho := r - 2ms_b^2$, $\gamma := -b$, $\beta := b - g$, $M_k := m$. Then, the metric can be written as

$$ds^2 = \rho^2(dt + B_t dt + B_\phi d\phi)^2 + \rho^{-1}ds^2_{(4)}, \tag{39}$$

where

$$\rho^2 = \frac{\varrho^3(\varrho + 2M_k(s_\beta^2 - s_\gamma^2)) + 2a^2x^2[\varrho^2 + M_k(s_\beta^2 - s_\gamma^2) - 2M_k^2 s_\gamma^2 c_\gamma^2(s_\gamma^2 + 2s_\beta c_\beta s_\gamma c_\gamma + s_\beta^2 c_\gamma^2)] + a^4x^4}{\varrho^2(\varrho - 2s_\gamma^2 M_k + 2s_\beta^2 M_k)^2}, \tag{40}$$

$$B_t = -2M_k a x \frac{[\varrho + 2M_k(s_\beta^2 - s_\gamma^2)][-2M_k s_\gamma^2 c_\gamma^2 (c_\beta s_\gamma + s_\beta s_\gamma) + (c_\beta s_\gamma^3 + s_\beta c_\gamma + s_\beta s_\gamma^2 c_\gamma)\varrho] + M_k a^2 x^2 (c_\beta s_\gamma^3 + s_\beta c_\gamma + s_\beta s_\gamma^2 c_\gamma)}{\varrho^3(\varrho + 2M_k(s_\beta^2 - s_\gamma^2)) + 2a^2x^2[\varrho^2 + M_k(s_\beta^2 - s_\gamma^2) - 2M_k^2 s_\gamma^2 c_\gamma^2 (s_\gamma^2 + 2s_\beta c_\beta s_\gamma c_\gamma + s_\beta^2 c_\gamma^2)] + a^4 x^4},$$

$$B_\phi = 2M_k s_\beta c_\beta x \frac{(\varrho - 2s_\gamma^2 M_k)(\varrho - 2c_\gamma^2 M_k) + a^2}{(\varrho - 2s_\gamma^2 M_k)(\varrho - 2c_\gamma^2 M_k) + a^2x^2} + B_t {\omega^0}_\phi, \tag{41}$$

$$\begin{aligned} ds_{(4)}^2 \;=\; & -\frac{\varrho^2 - 2M_k(c_\gamma^2 + s_\gamma^2)\varrho + 4M_k^2 c_\gamma^2 s_\gamma^2 + a^2x^2}{\rho^2[\varrho^2 + 2M_k(s_\beta^2 - s_\gamma^2)\varrho + a^2x^2]} \left(dt + {\omega^0}_\phi d\phi\right)^2 \\ & + \rho \frac{[\varrho^2 + 2M_k(s_\beta^2 - s_\gamma^2)\varrho + a^2x^2][(\varrho - 2s_\gamma^2 M_k)(\varrho - 2c_\gamma^2 M_k) + a^2]}{(\varrho - 2s_\gamma^2 M_k)(\varrho - 2c_\gamma^2 M_k) + a^2x^2} d\phi^2 \\ & + \rho\varrho[\varrho + 2M_k(s_\beta^2 - s_\gamma^2)] \left[\frac{d\varrho^2}{(\varrho - 2s_\gamma^2 M_k)(\varrho - 2c_\gamma^2 M_k) + a^2} + \frac{dx^2}{1-x^2}\right], \end{aligned} \tag{42}$$

$${\omega^0}_\phi = -2M_k a \frac{-\varrho(s_\beta s_\gamma^3 + c_\beta c_\gamma^3) + 2M_k s_\gamma^2 c_\gamma^2 (s_\beta s_\gamma + c_\beta s_\gamma)}{(\varrho - 2s_\gamma^2 M_k)(\varrho - 2c_\gamma^2 M_k) + a^2x^2}. \tag{43}$$

This exactly coincides with the metric of the *rotating* Ishihara-Matsuno solutions in Ref. [8].

VII. SUMMARY

In this paper, using the $SL(2,R)$-duality transformation that the reduced Lagrangian possesses upon reduction to four dimensions, we have succeeded in constructing general Kaluza-Klein black hole solutions in (the bosonic sector of) five-dimensional minimal supergravity, where we have used the electrically/magnetically charged boosted black string solution as a seed solution. Our solutions are the most general ones in the sense that in that theory, from a four-dimensional point of view, such a class of regular black hole solutions can be specified by six independent charges, its mass, angular momentum along four dimensions, electric and magnetic charges of the Maxwell fields in addition to the Kaluza-Klein electric and magnetic monopole charges. From the five-dimensional point of view, like known Kaluza-Klein charged black hole solutions, the black hole spacetime has two horizons, the outer and inner horizons, and although the cross-section geometry of the outer horizon is of S^3, at large distances the spacetime behaves effectively as a four-dimensional spacetime, which is due to the existence of a Kaluza-Klein monopole charge.

The present metric form of our solutions is considerably lengthy and complicated, which, as a result, makes us difficult to analyze the physical properties of our solutions. To do so, it should be written in terms of some physical parameters such as (M, J, Q, P, q, p) rather than the transformation parameters α, β, b, d, g. As a future work, we would like to present it in a more compact and physically clearer form.

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Appendix A: Coefficients

The coefficients $a_1, \cdots, a_8, b_1, \cdots, b_{10}, c_1, c_2, p_1, \cdots, p_{10}, q_1, \cdots, q_{10}, r_1, r_2$ in Eqs.(23)-(25) are given by, respectively,

$$a_1 = -12m^2 s_b s_d c_d^3 f \left[c_b\left(s_b^2 + c_b^2\right)s_g - 2s_b c_b^2 c_g f\right], \tag{A1}$$

$$a_2 = a_8 a^{-2} = -2mac_d^3\left[\left(s_b^2 + c_b^2\right)s_g + 2s_b c_b c_g f\right], \tag{A2}$$

$$a_3 = -8m^3 s_d c_d^3 \Big[s_d^2 s_g^2 c_g + s_b c_b s_g\left(s_d^2 + 3c_d^2 s_g^2\right) f + 4s_b^6 c_g \left(-3 - 4s_d^2 + \left(3 + 5s_d^2 + 6s_d^4\right)s_g^2\right) f^2$$

$$
\begin{aligned}
& +6s_b^4c_g\left(-2-4s_d^2+\left(3+5s_d^2+6s_d^4\right)s_g^2\right)f^2+2s_b^3c_bs_g\left(-6+9s_d^2+6s_d^4+2\left(6+19s_d^2+9s_d^4\right)s_g^2\right)f^3 \\
& +2s_b^2c_g\left(-4s_d^2+3\left(1+3\left(s_d^2+s_d^4\right)\right)s_g^2\right)f^2+4s_b^5c_bs_g\left(3+6s_g^2+s_d^4\left(3+9s_g^2\right)+s_d^2\left(9+19s_g^2\right)\right)f^3\Big],
\end{aligned} \tag{A3}
$$

$$
\begin{aligned}
a_4 \;=\; & -4m^2ac_d^3\Big[6s_bc_b\left(4+7s_d^2+3s_d^4\right)s_g^2c_gf^3+4s_b^3c_bc_g\left(1+3c_d^2s_g^2\right)f+s_g\left(s_g^2+3s_d^2c_g^2\right) \\
& +4s_b^4s_g\left(5+6s_g^2+3s_d^2\left(5+3s_d^2\right)c_g^2\right)f^2+6s_b^2s_g\left(2+4s_g^2+6s_d^4c_g^2+s_d^2\left(9+10s_g^2\right)\right)f^2\Big],
\end{aligned} \tag{A4}
$$

$$
\begin{aligned}
a_5 \;=\; & 16m^4s_b^2s_dc_d^3\Big[12s_b^5c_bc_d^4s_gf^3+12s_b^6c_d^4\left(2+3s_d^2\right)c_gf^4+s_bc_bs_d^2\left(17+9s_d^2\right)s_gc_g^2f^3+3s_d^4c_g^3f^2 \\
& +2s_b^3c_bs_g\left(6\left(2+s_g^2\right)+3s_d^4\left(4+3s_g^2\right)+s_d^2\left(27+19s_g^2\right)\right)f^3+2s_b^4c_g(12c_g^2+18s_d^6\left(2+s_g^2\right) \\
& +s_d^2\left(59+29s_g^2\right)+s_d^4\left(84+39s_g^2\right))f^4+s_b^2c_g\left(24s_g^2+9s_d^6\left(5+4s_g^2\right)+s_d^2\left(34+42s_g^2\right)+s_d^4\left(84+66s_g^2\right)\right)f^4\Big]
\end{aligned} \tag{A5}
$$

$$
\begin{aligned}
a_6 \;=\; & -8m^3as_b^2c_d^3\Big[3s_d^2\left(4+15s_d^2+9s_d^4\right)s_gc_g^2f^4+2s_bc_bc_g\left(3s_d^2+\left(4+21s_d^2+9s_d^4\right)s_g^2\right)f^3+4s_b^3c_bc_g(2+3s_d^2 \\
& +\left(8+21s_d^2+9s_d^4\right)s_g^2)f^3+4s_b^4s_g\left(3+4s_g^2+9s_d^2c_d^2c_g^2\right)f^2+4s_b^2s_g\left(2+3s_g^2+3s_d^2\left(2+3s_d^2\right)c_g^2\right)f^2\Big],
\end{aligned} \tag{A6}
$$

$$
\begin{aligned}
a_7 \;=\; & 4m^2a^2s_dc_d^3\Big[s_d^2c_g\left(1+2s_g^2\right)+s_bc_bs_g\left(3+4s_d^2+6c_d^2s_g^2\right)f+2s_b^3c_bs_g\left(3+4s_d^2+6c_d^2s_g^2\right)f \\
& +2s_b^2c_g\left(3+8s_d^2+6s_d^4+6\left(1+3s_d^2+2s_d^4\right)s_g^2\right)f^2+2s_b^4c_g\left(3+8s_d^2+6s_d^4+6\left(1+3s_d^2+2s_d^4\right)s_g^2\right)f^2\Big],
\end{aligned} \tag{A7}
$$

$$
b_1 \;=\; b_6a^{-2}=2mas_d\left[-2s_bc_b\left(3+s_d^2\right)s_gc_gf-\left(c_b^2+s_b^2\right)s_d^2\left(1+2s_g^2\right)\right], \tag{A8}
$$

$$
\begin{aligned}
b_2 \;=\; & -4m^2as_d\Big[4s_b^3c_bs_gc_g\left(6+13s_d^2+3s_d^4+\left(3+14s_d^2+3s_d^4\right)s_g^2\right)f \\
& +2s_b^4\left(-3+10s_d^2+33s_d^4+18s_d^6+2\left(6+22s_d^2+51s_d^4+27s_d^6\right)s_g^2+4\left(3+5s_d^2+15s_d^4+9s_d^6\right)s_g^4\right)f^2 \\
& +s_b^2\left(-6+8s_d^2+57s_d^4+36s_d^6+2\left(12+32s_d^2+93s_d^4+54s_d^6\right)s_g^2+8\left(3+5s_d^2+15s_d^4+9s_d^6\right)s_g^4\right)f^2 \\
& +s_d^2c_g^2\left(2s_g^2+s_d^2\left(3+6s_g^2\right)\right)+s_bc_bs_gc_g\left(3+6s_g^2+6s_d^4c_g^2+s_d^2\left(23+28s_g^2\right)\right)f\Big],
\end{aligned} \tag{A9}
$$

$$
b_3 \;=\; -2ma^2\left[\left(1+3s_d^2\right)s_gc_g+2s_b^2\left(1+3s_d^2\right)s_gc_g+2s_bc_b\left(1+2s_g^2\right)f\right], \tag{A10}
$$

$$
\begin{aligned}
b_4 \;=\; & -8m^3as_df^4\Big[s_d^2f^{-4}c_g^2\left(3s_d^4+\left(1+3s_d^2+6s_d^4\right)s_g^2\right) \\
& +12s_b^5c_bc_d^4f^{-1}s_gc_g\left(7+24s_d^2+9s_d^4+2\left(6+19s_d^2+9s_d^4\right)s_g^2\right) \\
& +s_bc_bf^{-3}s_gc_g\left(s_d^2+24s_d^4+9s_d^6+\left(3+14s_d^2+45s_d^4+18s_d^6\right)s_g^2\right) \\
& +12s_b^6c_d^4f^{-2}\left(-1+5s_g^2+6s_g^4+6s_d^4\left(1+3s_g^2+2s_g^4\right)+s_d^2\left(2+11s_g^2+10s_g^4\right)\right) \\
& +2s_b^4f^{-2}(-6+33s_g^2+42s_g^4+54s_d^8\left(1+3s_g^2+2s_g^4\right)+9s_d^6\left(12+41s_g^2+30s_g^4\right) \\
& +3s_d^4\left(16+101s_g^2+94s_g^4\right)+s_d^2\left(-10+145s_g^2+178s_g^4\right)) \\
& +4s_b^3c_bf^{-1}s_gc_g\left(-3+24s_g^2+s_d^2\left(44+121s_g^2+3s_d^2\left(47+39s_d^2+9s_d^4+\left(88+71s_d^2+18s_d^4\right)s_g^2\right)\right)\right) \\
& +s_b^2f^{-2}(6\left(s_g^2+2s_g^4\right)+54s_d^8\left(1+3s_g^2+2s_g^4\right)+6s_d^6\left(15+53s_g^2+39s_g^4\right)+2s_d^2\left(-4+43s_g^2+58s_g^4\right) \\
& +3s_d^4\left(7+66\left(s_g^2+s_g^4\right)\right))\Big],
\end{aligned} \tag{A11}
$$

$$
\begin{aligned}
b_5 \;=\; & 4a^2m^2f^3\Big[f^{-3}s_gc_g\left(3s_d^2\left(-1-2s_d^2+s_d^4\right)+\left(-1-6s_d^2-9s_d^4+4s_d^6\right)s_g^2\right) \\
& +2s_b^2f^{-1}s_gc_g\left(3\left(-2-11s_d^2-25s_d^4-4s_d^6+6s_d^8\right)+4\left(-4-9s_d^2-24s_d^4-5s_d^6+6s_d^8\right)s_g^2\right) \\
& +2s_bc_bf^{-2}\left(3s_d^4+3\left(-1-3s_d^2+5s_d^4+s_d^6\right)s_g^2+4\left(-1-3s_d^2+3s_d^4+s_d^6\right)s_g^4\right) \\
& +4s_b^3c_bf^{-2}\left(-1+3s_d^4+\left(-5+3s_d^2\left(-3+5s_d^2+s_d^4\right)\right)s_g^2+4\left(-1-3s_d^2+3s_d^4+s_d^6\right)s_g^4\right)
\end{aligned}
$$

$$+4s_b^4 f^{-1} s_g c_g \left(-5-8s_g^2-6s_d^2\left(4+3s_g^2\right)+3s_d^8\left(3+4s_g^2\right)-2s_d^6\left(3+5s_g^2\right)-6s_d^4\left(7+8s_g^2\right)\right)\Big], \tag{A12}$$

$$\begin{aligned} b_7 \;=\; & -16am^4 s_d f^5 \Big[s_b c_b s_d^6 f^{-4}\left(9+5s_d^2\right) s_g c_g^3 + s_d^8 f^{-5} c_g^4 + 4s_b^7 c_b c_d^6 f^{-2} s_g c_g (3+31s_d^2+30s_d^4 \\ & +6\left(3+8s_d^2+5s_d^4\right)s_g^2) + s_b^3 c_b s_d^2 f^{-2} s_g c_g (-17+78s_d^2+318s_d^4+309s_d^6+90s_d^8+(27+150s_d^2 \\ & +357s_d^4+316s_d^6+90s_d^8)s_g^2) + 12s_b^5 c_b c_d^4 f^{-2} s_g c_g (-2+3s_g^2+15s_d^6 c_g^2 + s_d^2\left(5+14s_g^2\right)+s_d^4\left(26+30s_g^2\right)) \\ & +s_b^2 s_d^4 f^{-3} c_g^2\left(-3+27s_g^2+24s_d^6 c_g^2+3s_d^2\left(4+15s_g^2\right)+s_d^4\left(41+50s_g^2\right)\right) \\ & +4s_b^8 c_d^6 f^{-1}\left(-6+24s_g^2+36s_g^4+36s_d^6 c_g^4+6s_d^4 c_g^2\left(7+16s_g^2\right)+s_d^2\left(1+92s_g^2+96s_g^4\right)\right) \\ & +2s_b^6 c_d^4 f^{-1}(-12+36s_g^2+72s_g^4+144s_d^8 c_g^4+6s_d^6 c_g^2\left(43+70s_g^2\right)+s_d^2\left(-35+287s_g^2+354s_g^4\right) \\ & +s_d^4\left(82+605s_g^2+534s_g^4\right)) + s_b^4 f^{-1}(-24s_g^2+s_d^2(-34+142s_g^2+228s_g^4+216s_d^{10}c_g^4+18s_d^8 c_g^2\left(41+50s_g^2\right) \\ & +6s_d^2\left(-8+149s_g^2+164s_g^4\right)+3s_d^4\left(102+661s_g^2+564s_g^4\right)+s_d^6\left(843+2469s_g^2+1628s_g^4\right)))\Big], \end{aligned} \tag{A13}$$

$$\begin{aligned} b_8 \;=\; & 8a^2m^3f^4\Big[s_d^6\left(-1+3s_d^2\right)f^{-4}s_g c_g^3 + 2s_b c_b s_d^4 f^{-3} c_g^2\left(2s_d^2+\left(-3+11s_d^2+6s_d^4\right)s_g^2\right) \\ & +s_b^2 s_d^2 f^{-2} s_g c_g\left(-3-6s_d^2-14s_d^4+63s_d^6+54s_d^8+\left(-15+30s_d^2+37s_d^4+78s_d^6+54s_d^8\right)s_g^2\right) \\ & +4s_b^5 c_b c_d^4 f^{-1}\left(-2+s_d^2+12s_d^4+3\left(-4-7s_d^2+21s_d^4+12s_d^6\right)s_g^2+6\left(-2-3s_d^2+9s_d^4+6s_d^6\right)s_g^4\right) \\ & +2s_b^3 c_b f^{-1}(s_d^2\left(-3+12s_d^2+41s_d^4+24s_d^6\right)+\left(-4+3s_d^2\left(-13-5s_d^2+75s_d^4+87s_d^6+24s_d^8\right)\right)s_g^2 \\ & +8\left(-1+s_d^2\left(-3+s_d^4\left(5+3s_d^2\right)^2\right)\right)s_g^4) + 12s_b^6 c_d^4 f^{-2} s_g c_g\left(-1-2s_d^2-3s_d^4-2s_g^2+6s_d^6 c_g^2\right) \\ & +4s_b^4 f^{-2} s_g c_g\left(-2-5s_g^2+s_d^2\left(-6-9s_g^2+15s_d^2\left(-1+s_g^2\right)+27s_d^8 c_g^2+12s_d^6\left(3+4s_g^2\right)+s_d^4\left(-6+32s_g^2\right)\right)\right)\Big], \end{aligned} \tag{A14}$$

$$\begin{aligned} b_9 \;=\; & 4a^3m^2s_d f^2\Big[s_d^2 f^{-2} c_g^2\left(1+\left(2+6s_d^2\right)s_g^2\right)+2s_b^4(3+4s_d^2+3s_d^4+2\left(3+7s_d^2+30s_d^4+18s_d^6\right)s_g^2 \\ & +4\left(3+5s_d^2+15s_d^4+9s_d^6\right)s_g^4)+s_b^2(6+12s_d^2+9s_d^4+6\left(2+6s_d^2+21s_d^4+12s_d^6\right)s_g^2 \\ & +8\left(3+5s_d^2+15s_d^4+9s_d^6\right)s_g^4)+4s_b^3 c_b f^{-1} s_g c_g\left(3s_g^2+3s_d^4\left(-1+s_g^2\right)+s_d^2\left(1+14s_g^2\right)\right) \\ & +s_b c_b f^{-1} s_g c_g\left(3+6s_g^2+6s_d^4\left(-1+s_g^2\right)+s_d^2\left(3+28s_g^2\right)\right)\Big], \end{aligned} \tag{A15}$$

$$b_{10} = -2a^4m\left[\left(1+3s_d^2\right)s_g c_g+2s_b^2\left(1+3s_d^2\right)s_g c_g+2s_b c_b\left(1+2s_g^2\right)f\right], \tag{A16}$$

$$c_2 = -2m\left[\left(1+3s_d^2\right)s_g c_g+2s_b^2\left(1+3s_d^2\right)s_g c_g+2s_b c_b\left(1+2s_g^2\right)f\right] \tag{A17}$$

$$p_1 = -4\sqrt{3}ms_b c_b c_d^2 f, \tag{A18}$$

$$\begin{aligned} p_2 \;=\; & -4\sqrt{3}m^2c_d^2\Big[s_d^2 s_g c_g+4s_b^2\left(2+5s_d^2+3s_d^4\right)s_g c_g f^2 \\ & +4s_b^4\left(2+5s_d^2+3s_d^4\right)s_g c_g f^2+2s_b^3 c_b c_d^2\left(3+2s_g^2\right)f+s_b c_b\left(3s_d^2+2c_d^2 s_g^2\right)f\Big], \end{aligned} \tag{A19}$$

$$p_3 = -2\sqrt{3}ma\left(c_b^2+s_b^2\right)s_d c_d^2, \tag{A20}$$

$$\begin{aligned} p_4 \;=\; & -8\sqrt{3}m^3c_d^2\Big[s_d^4 s_g c_g+6s_b^2 s_d^2\left(2+5s_d^2+3s_d^4\right)s_g c_g f^2+8s_b^6\left(2+6s_d^2+7s_d^4+3s_d^6\right)s_g c_g f^2 \\ & +4s_b^4 c_d^2\left(4+9\left(s_d^2+s_d^4\right)\right)s_g c_g f^2+s_b c_b s_d^2\left(s_d^2+3c_d^2 s_g^2\right)f+4s_b^5 c_b c_d^2\left(6+10s_d^2+3s_d^4+\left(8+19s_d^2+9s_d^4\right)s_g^2\right)f^3 \\ & +2s_b^3 c_b c_d^2\left(8s_g^2+6s_d^4\left(1+3s_g^2\right)+s_d^2\left(11+32s_g^2\right)\right)f^3\Big], \end{aligned} \tag{A21}$$

$$p_5 = -4\sqrt{3}m^2as_d c_d^2\left[\left(c_b^2+s_b^2\right)\left(s_d^2+2s_b^2c_d^2\right)+c_d^2\left(1+12s_b^2c_b^2c_d^2f^2\right)s_g^2\right], \tag{A22}$$

$$p_6 = -4\sqrt{3}ma^2s_b c_b c_d^2 f, \tag{A23}$$

$$\begin{aligned} p_7 \;=\; & -16\sqrt{3}m^4s_b^3c_d^4f^4\Big[4s_b^5c_d^2\left(8+15s_d^2+9s_d^4\right)s_g c_g+s_b s_d^2\left(20+24s_d^2+9s_d^4\right)s_g c_g \\ & +4s_b^3\left(8+24s_d^2+24s_d^4+9s_d^6\right)s_g c_g+3c_b s_d^4 f^{-1}c_g^2+4s_b^4 c_b f^{-1}\left(2+5s_d^2+3s_d^4\right)\left(1+2s_g^2\right) \end{aligned}$$

$$+2s_b^2c_bf^{-1}\left(4s_g^2+s_d^4\left(6+9s_g^2\right)+s_d^2\left(5+11s_g^2\right)\right)\Big], \tag{A24}$$

$$p_8 = -8\sqrt{3}m^3as_b^2s_dc_d^4f^3\Big[8s_bc_b^3s_gc_g+3s_d^2f^{-1}c_g^2+4s_b^2f^{-1}\left(1+3s_d^2c_g^2\right)+4s_b^4f^{-1}\left(2+s_g^2+3s_d^2c_g^2\right)\Big], \tag{A25}$$

$$p_9 = 4\sqrt{3}m^2a^2c_d^2f^2\Big[s_d^4f^{-2}s_gc_g+4s_b^2\left(-1+4s_d^4+3s_d^6\right)s_gc_g+4s_b^4\left(-1+4s_d^4+3s_d^6\right)s_gc_g$$
$$+2s_b^3c_bc_d^2f^{-1}\left(-1+2s_d^2s_g^2\right)+s_bc_bs_d^2f^{-1}\left(-1+2c_d^2s_g^2\right)\Big], \tag{A26}$$

$$p_{10} = -2\sqrt{3}ma^3\left(c_b^2+s_b^2\right)s_dc_d^2, \tag{A27}$$

$$q_1 = -2\sqrt{3}amc_df\left[\left(1+2s_b^2\right)s_d^2f^{-1}s_g+2s_bc_bc_g\right], \tag{A28}$$

$$q_2 = -4\sqrt{3}am^2c_df^2\Big[s_d^2\left(4+15s_d^2+9s_d^4\right)s_gc_g^2+2s_b^4s_g\left(4+17s_d^2+37s_d^4+18s_d^6+2\left(2+4s_d^2+15s_d^4+9s_d^6\right)s_g^2\right)$$
$$+s_b^2s_g\left(8+22s_d^2+65s_d^4+36s_d^6+4\left(2+4s_d^2+15s_d^4+9s_d^6\right)s_g^2\right)+s_bc_bf^{-1}c_g\left(-2s_d^4+2s_g^2+s_d^2\left(3+10s_g^2\right)\right)$$
$$+2s_b^3c_bf^{-1}c_g\left(3-2s_d^4+2s_g^2+s_d^2\left(3+10s_g^2\right)\right)\Big], \tag{A29}$$

$$q_3 = -2\sqrt{3}a^2ms_dc_df\left[-2s_bc_bs_g+f^{-1}c_g+2s_b^2f^{-1}c_g\right], \tag{A30}$$

$$q_4 = -8\sqrt{3}am^3c_df^3\Big[s_d^4f^{-1}\left(4+11s_d^2+6s_d^4\right)s_gc_g^2+4s_b^5c_bc_d^2c_g\left(6+10s_d^2-6s_d^4-9s_d^6+\left(8+39s_d^2+33s_d^4+6s_d^6\right)s_g^2\right)$$
$$+s_b^2s_d^2f^{-1}s_g\left(12+31s_d^2+68s_d^4+36s_d^6+2\left(9+19s_d^2+35s_d^4+18s_d^6\right)s_g^2\right)$$
$$+2s_b^4f^{-1}s_g\left(8+22s_d^2+50s_d^4+78s_d^6+36s_d^8+\left(8+31s_d^2+60s_d^4+81s_d^6+36s_d^8\right)s_g^2\right)$$
$$+s_bc_bs_d^2f^{-2}c_g\left(3s_g^2+s_d^4\left(-3+2s_g^2\right)+s_d^2\left(1+11s_g^2\right)\right)+4s_b^6c_d^2f^{-1}s_g\left(4c_g^2+9s_d^2c_g^2+12s_d^6c_g^2+s_d^4\left(20+21s_g^2\right)\right)$$
$$+2s_b^3c_bc_g\left(8s_g^2+s_d^2\left(11+7s_d^2-24s_d^4-18s_d^6+6c_d^2\left(8+11s_d^2+2s_d^4\right)s_g^2\right)\right)\Big], \tag{A31}$$

$$q_5 = 4\sqrt{3}a^2m^2s_dc_df^2\Big[2s_bc_bf^{-1}s_g\left(7s_d^2+\left(-1+8s_d^2+s_d^4\right)s_g^2\right)+4s_b^3c_bf^{-1}s_g\left(1+7s_d^2+\left(-1+8s_d^2+s_d^4\right)s_g^2\right)$$
$$+f^{-2}c_g\left(s_d^2\left(-1+s_d^2\right)+\left(-1-3s_d^2+2s_d^4\right)s_g^2\right)+2s_b^2c_g\left(-4-13s_d^2+2s_d^4+6s_d^6+2\left(1-14s_d^2-s_d^4+6s_d^6\right)s_g^2\right)$$
$$+4s_b^4c_g\left(-4+s_g^2+s_d^2\left(-8+s_d^2+3s_d^4+\left(-14-s_d^2+6s_d^4\right)s_g^2\right)\right)\Big], \tag{A32}$$

$$q_6 = -2\sqrt{3}a^3mc_df\left[\left(1+2s_b^2\right)s_d^2f^{-1}s_g+2s_bc_bc_g\right], \tag{A33}$$

$$q_7 = 16\sqrt{3}am^4s_bc_df^4\Big[s_bs_d^6f^{-2}\left(7+4s_d^2\right)s_gc_g^2+c_bs_d^8f^{-3}c_g^3$$
$$+s_b^3s_d^2s_g\left(-20+18s_d^2+148s_d^4+149s_d^6+42s_d^8+\left(-8+38s_d^2+159s_d^4+151s_d^6+42s_d^8\right)s_g^2\right)$$
$$-4s_b^7c_d^6s_g\left(8+5s_d^2+6s_d^4+2\left(4+3\left(s_d^2+s_d^4\right)\right)s_g^2\right)+4s_b^6c_bc_d^6f^{-1}c_g\left(-2-4s_g^2+6s_d^4c_g^2-s_d^2\left(1+6s_g^2\right)\right)$$
$$+s_b^2c_bs_d^4f^{-1}c_g\left(-3+13s_g^2+18s_d^6c_g^2+s_d^2\left(11+29s_g^2\right)+s_d^4\left(33+38s_g^2\right)\right)$$
$$+2s_b^5c_d^2s_g\left(-16c_g^2+12s_d^8c_g^2+s_d^4\left(11+7s_g^2\right)-4s_d^2\left(9+10s_g^2\right)+s_d^6\left(40+39s_g^2\right)\right)$$
$$+2s_b^4c_bc_d^2f^{-1}c_g\left(-4s_g^2+s_d^2\left(-5-3s_g^2+3s_d^2\left(s_g^2+2s_d^2f^{-2}c_g^2\right)\right)\right)\Big], \tag{A34}$$

$$q_8 = 8\sqrt{3}a^2m^3s_dc_df^3\Big[2s_bc_bs_d^4\left(4+s_d^2\right)f^{-2}s_gc_g^2+s_d^6f^{-3}c_g^3$$
$$+s_b^2s_d^2f^{-1}c_g\left(-3+s_d^2+27s_d^4+18s_d^6+\left(17+11s_d^2+26s_d^4+18s_d^6\right)s_g^2\right)$$
$$+2s_b^3c_bs_g\left(-4+15s_d^2+77s_d^4+64s_d^6+12s_d^8+\left(4+21s_d^2+76s_d^4+63s_d^6+12s_d^8\right)s_g^2\right)$$
$$+4s_b^6c_d^2f^{-1}c_g\left(-2+3s_g^2+6s_d^6c_g^2-s_d^2\left(4+s_g^2\right)+s_d^4\left(5+6s_g^2\right)\right)$$
$$+4s_b^5c_bs_d^2c_d^2s_g\left(23+25s_g^2+6s_d^4c_g^2+s_d^2\left(26+27s_g^2\right)\right)$$
$$+4s_b^4f^{-1}c_g\left(-1+4s_g^2+s_d^2\left(-5+10s_g^2+s_d^2\left(1+8s_g^2+3s_d^2\left(5+3s_d^2\right)c_g^2\right)\right)\right)\Big], \tag{A35}$$

$$q_9 = 4\sqrt{3}a^3m^2c_df^2\Big[2s_d^4f^{-2}s_gc_g^2 - s_bc_bs_d^2f^{-1}c_g\left(1-6s_g^2+2s_d^2c_g^2\right) - 2s_b^3c_bf^{-1}c_g\left(1+s_d^2\left(1-6s_g^2\right)+2s_d^4c_g^2\right)$$
$$+2s_b^4s_g\left(-2+12s_d^6c_g^2-s_d^2\left(5+6s_g^2\right)+s_d^4\left(15+14s_g^2\right)\right)$$
$$+s_b^2s_g\left(-4+24s_d^6c_g^2-6s_d^2\left(1+2s_g^2\right)+s_d^4\left(33+28s_g^2\right)\right)\Big], \tag{A36}$$

$$q_{10} = r_2a^4 = -2\sqrt{3}a^4ms_dc_df\left[-2s_bc_bs_g+f^{-1}c_g+2s_b^2f^{-1}c_g\right]. \tag{A37}$$

(A38)

Appendix B: D=4 SL(2,R) duality

In this section, we summarize the results of the solution-generation-technique [10] using the $SL(2,R)$ duality symmetry [2] of five-dimensional minimal supergravity dimensionally reduced to four dimensions. Since this solution-generation method is already described [10] in detail, we will be brief.

The Lagrangian is

$$\mathcal{L} = E^{(5)}\left(R^{(5)} - \frac{1}{4}F_{MN}F^{MN}\right) - \frac{1}{12\sqrt{3}}\epsilon^{MNPQR}F_{MN}F_{PQ}A_R. \tag{B1}$$

$M, N, \ldots$ are five-dimensional curved indices running over $0,1,2,3$ and 5. $E^{(5)}$ is the determinant of the vielbein

$$E^{(5)A}_{\ \ M} = \begin{pmatrix} \rho^{-\frac{1}{2}}E^{(4)\alpha}_{\ \ \mu} & B_\mu\rho \\ 0 & \rho \end{pmatrix} \tag{B2}$$

of the metric $G^{(5)}_{MN} = E^{(5)A}_{\ \ M}E^{(5)B}_{\ \ N}\eta_{AB}$, $\eta_{AB} \equiv \mathrm{diag}(-1,+1,+1,+1,+1)$. x^μ ($\mu = 0,1,2,3$) is the four-dimensional coordinates. We take $\partial/\partial x^5$ as the Killing vector.

After the dimensional reduction and dualizing the gauge field A_μ, we end up with a four-dimensional $SL(2,R)/U(1)$ non-linear sigma model coupled to two $U(1)$ gauge fields and gravity:

$$\begin{aligned}
\mathcal{L} + \mathcal{L}_{\text{Lag.mult.}} &= E^{(4)}R^{(4)} + \mathcal{L}_S + \mathcal{L}_V \quad \text{(up to a complete square)},\\
\mathcal{L}_S &\equiv -E^{(4)}\left(\frac{3}{2}\partial_\mu\ln\rho\,\partial^\mu\ln\rho + \frac{1}{2}\rho^{-2}\partial_\mu A_5\partial^\mu A_5\right),\\
\mathcal{L}_V &\equiv -\frac{1}{4}E^{(4)}\mathcal{G}^T_{\mu\nu}N^{\mu\nu\rho\sigma}\mathcal{G}_{\rho\sigma}.
\end{aligned} \tag{B3}$$

Here $N^{\mu\nu\rho\sigma}$ is given by

$$\begin{aligned}
N^{\mu\nu\rho\sigma} &= m\ 1^{\mu\nu\rho\sigma} + a\ (*)^{\mu\nu\rho\sigma},\\
V^{-1}mV^{-1} &= K - \frac{1}{2}(\Phi\Phi^*K + K\Phi^*\Phi) + \frac{1}{4}\Phi\Phi^{*2}K\Phi,\\
V^{-1}aV^{-1} &= -\Phi^*K - \Phi + \frac{1}{2}(\Phi\Phi^{*2}K + K\Phi^{*2}\Phi) + \frac{1}{3}\Phi\Phi^*\Phi - \frac{1}{4}\Phi\Phi^{*3}K\Phi,
\end{aligned} \tag{B4}$$

where $1^{\mu\nu\rho\sigma} \equiv \frac{1}{2}\left(G^{(4)\mu\rho}G^{(4)\nu\sigma} - G^{(4)\nu\rho}G^{(4)\mu\sigma}\right)$, $(*)^{\mu\nu\rho\sigma} \equiv \frac{1}{2}E^{(4)-1}\epsilon^{\mu\nu\rho\sigma}$, with

$$V \equiv \begin{pmatrix} \rho^{-\frac{1}{2}} & 0 \\ 0 & \rho^{\frac{3}{2}} \end{pmatrix},\quad \Phi \equiv \begin{pmatrix} 0 & \sqrt{3}\phi \\ \sqrt{3}\phi & 0 \end{pmatrix},\quad \Phi^* \equiv \begin{pmatrix} 2\phi & 0 \\ 0 & 0 \end{pmatrix},\quad K \equiv (1+\Phi^{*2})^{-1},\ \phi \equiv \frac{1}{\sqrt{3}}\rho^{-1}A_5. \tag{B5}$$

The two-component vector $\mathcal{G}_{\mu\nu} = \begin{pmatrix} \tilde{A}_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix}$ contains of the field strengths of two $U(1)$ gauge fields $\tilde{A}_\mu$ and B_μ, where $\tilde{A}_\mu$ is a dual of A_μ. Their relation is

$$\tilde{A}_{\mu\nu} = \rho(*F^{(4)})_{\mu\nu} - \frac{2}{\sqrt{3}}A_5F^{(4)}_{\mu\nu} + \frac{1}{\sqrt{3}}A_5^2B_{\mu\nu} \tag{B6}$$

with $F^{(4)}_{\mu\nu} \equiv F'_{\mu\nu} + B_{\mu\nu}A_5$, $F'_{\mu\nu} \equiv \partial_\mu A'_\nu - \partial_\nu A'_\mu$ and $A'_\mu \equiv A_\mu - B_\mu A_5$.

Let $\mathcal{F}_{\mu\nu} \equiv \begin{pmatrix} \mathcal{G}_{\mu\nu} \\ \mathcal{H}_{\mu\nu} \end{pmatrix}$ be a four-component field strength vector consisting of $\mathcal{G}_{\mu\nu}$ and $\mathcal{H}_{\mu\nu} \equiv \begin{pmatrix} \mathcal{H}^{\tilde{A}}_{\mu\nu} \\ \mathcal{H}^{B}_{\mu\nu} \end{pmatrix} \equiv m(*\mathcal{G})_{\mu\nu} - a\,\mathcal{G}_{\mu\nu}$. Also let $\mathcal{V}_-$, $\mathcal{V}_+$ be scalar-field dependent four-by-four matrices

$$\mathcal{V}_+ = \begin{pmatrix} V & \\ & V^{-1} \end{pmatrix}, \quad \mathcal{V}_- = \exp \begin{pmatrix} & -\Phi^* \\ -\Phi & \end{pmatrix}. \tag{B7}$$

Then it was shown [2] that all the equations of motion and the Bianchi identity are invariant under the $SL(2,R)$ transformation

$$\mathcal{F}_{\mu\nu} \ \mapsto \ \Lambda^{-1}\mathcal{F}_{\mu\nu}, \tag{B8}$$

$$(\mathcal{V}_-\mathcal{V}_+)^T\mathcal{V}_-\mathcal{V}_+ \ \mapsto \ \Lambda^T(\mathcal{V}_-\mathcal{V}_+)^T\mathcal{V}_-\mathcal{V}_+\Lambda, \tag{B9}$$

where Λ is an $SL(2,R)$ group element generated by

$$E' = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \end{pmatrix}, \quad F' = \begin{pmatrix} 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } H' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \tag{B10}$$

Using this $SL(2,R)$ invariance, one can obtain a new solution by acting this $SL(2,R)$ transformation on various fields of a known solution. In this paper, Λ is taken to be

$$\Lambda \ = \ e^{-\alpha E'}e^{-\beta F'}, \tag{B11}$$

then the transformation rules are given by the following formulas:

$$\rho^{new} = \frac{\rho}{\left(1+\alpha\beta+\beta\frac{A_5}{\sqrt{3}}\right)^2+\beta^2\rho^2}, \tag{B12}$$

$$A_5^{new} = \sqrt{3}\frac{\left(\alpha+\frac{A_5}{\sqrt{3}}\right)\left(1+\alpha\beta+\beta\frac{A_5}{\sqrt{3}}\right)+\beta\rho^2}{\left(1+\alpha\beta+\beta\frac{A_5}{\sqrt{3}}\right)^2+\beta^2\rho^2}, \tag{B13}$$

$$B_\mu^{new} = \sqrt{3}\beta^2\gamma\tilde{A}_\mu + \left(\gamma^3+\sqrt{3}\beta\gamma^2A_5\right)B_\mu - \sqrt{3}\beta\gamma^2A_\mu + \beta^3\tilde{B}_\mu, \tag{B14}$$

$$\begin{aligned} A_\mu^{new} \ = \ & \left[\sqrt{3}\beta^2\gamma A_5^{new} - \beta(2+3\alpha\beta)\right]\tilde{A}_\mu + \left[-\sqrt{3}\alpha\gamma^2 + \gamma^3A_5^{new} - A_5((1+4\alpha\beta+3\alpha^2\beta^2) - \sqrt{3}\beta\gamma^2A_5^{new}))\right]B_\mu \\ & + \left[(1+4\alpha\beta+3\alpha^2\beta^2) - \sqrt{3}\beta\gamma^2A_5^{new})\right]A_\mu + \left[-\sqrt{3}\beta^2+\beta^3A_5^{new}\right]\tilde{B}_\mu, \end{aligned} \tag{B15}$$

where $\tilde{A}_\mu$ and $\tilde{B}_\mu$ are "vector potentials" of $\tilde{A}_{\mu\nu}$ and $\mathcal{H}^B_{\mu\nu}$ satisfying

$$(d\tilde{A})_{\mu\nu} = \tilde{A}_{\mu\nu}, \quad (d\tilde{B})_{\mu\nu} = \mathcal{H}^B_{\mu\nu}, \tag{B16}$$

respectively. $E^{(4)}$ remains unchanged through this transformation.

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